LP$^+$, K3$^+$, FDE$^+$, and their 'classical collapse'

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1   Introduction

In [7] I made explicit a 'collapse result' for the (multiple-conclusion) propositional logic LP$^+$, and left implicit the corresponding result for the dual (Strong Kleene) logic K3$^+$.[1] Such logics are philosophically motivated by paradoxical phenomena – cases of apparent 'over-determinacy' (gluts) or 'under-determinacy' (gaps). Such logics are also notoriously weak. But what the collapse results make plain is the sense in which such logics 'collapse' into (the stronger) classical logic in the absence of (let us say) over-/under-determinacy. Such collapse results, I argued in [7], carry philosophical interest: they illuminate a natural way of responding to the weakness of such logics.

Feedback on such ideas has convinced me that the first-order case is worth formulating. My aim here is to do just that: namely, explicitly record the first-order result(s) – including K3$^+$, and FDE$^+$. Though I give a few remarks on the topic at the end of the note, my aim here is not to further expound the driving philosophical interest in such results; that is a matter for a much larger project [8]. My aim here is to be as concise as possible without being cryptic. I rely on the setup (and proofs) in [7], and focus on the LP$^+$ case.

2   First-order LP$^+$

LP [3, 4, 22] is dual to K3 [18], both sublogics of classical logic. Following [7] I focus on the model-theoretic account of LP$^+$.[2] Moreover, I sacrifice (widely available) details for the sake of brevity and clarity. Priest [24] provides full discussion of the first-order model theory of LP (and an adequate tableau system) – and of the dual K3 logic – and readers not familiar with details are encouraged to consult Priest's given work.

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[1] Superscript '+' is used for the multiple-conclusion generalization of the corresponding single-conclusion logic. In [7, Appendix] I presented an adequate 2-sided sequent system for LP$^+$. The first formulation of this (propositional) system in print was due to Arnon Avron [5], to whom I'm grateful for correspondence on this point.

[2] I briefly sketch an adequate sequent system in the appendix.

2.1 LP validity

We assume a standard first-order syntax (though, for simplicity, we ignore function signs and identity), taking $\exists$ and $\neg$ and $\vee$ as our primtive connectives (defining $\forall$ and $\wedge$ and $\rightarrow$ in the usual way).

Our interpretations are structures $I = \langle D, d \rangle$, where $D \neq \emptyset$ and d assigns each constant an element of D and assigns each n-ary predicate a (total) function – the so-called *intension* of the predicate – from D^n into $V = \{1, .5, 0\}$. Combined with variable assignments ν, we have a denotation function δ defined in the familiar way: $\delta(t) = \nu(t)$ if t is variable; otherwise, $\delta(t) = d(t)$.

With such structures in hand, we define the notion of *semantic value* in terms of *valuations*, which are (total) functions from wff (standardly defined) and variable assignments into V. Leaving reference to the given structures implicit, the semantic value $|A|_\nu$ of wff A (relative to variable assignment ν) is defined along standard (many-valued) lines, where A and B are any wff:

1. Atomic: $|Pt_0t_1 \ldots t_n|_\nu = d(P)(\langle \delta(t_0), \delta(t_1), \ldots, \delta(t_n) \rangle)$.

2. Negation: $|\neg A|_\nu = 1 - |A|_\nu$.

3. Disjunction: $|A \vee B|_\nu = \max\{|A|_\nu, |B|_\nu\}$.

4. Quantifier: $|\exists x A|_\nu = \max\{|A|_{\nu^{[x]}} : \text{for each } x\text{-variant } \nu^{[x]} \text{ of } \nu\}$.[3]

Definition (LP model). *An LP model M is a pair $\langle I, \nu \rangle$, where I is an LP interpretation and ν a variable assignment.*

Definition (LP satisfaction). *Let A be any wff and X any set of wff. An LP model M satisfies A iff $|A|_\nu \in \{1, .5\}$; and M satisfies X iff M satisfies each element of X.*[4]

Definition (LP dissatisfaction). *Let A be any wff and X any set of wff. An LP model M dissatisfies A iff $|A|_\nu = 0$; and dissatisfies X iff M dissatisfies each element of X.*

Finally, we define *LP validity* $\models_{\text{lp}} \subseteq \wp(S) \times S$ per the usual recipe:[5]

Definition (LP). *$X \models_{\text{lp}} A$ iff no LP model satisfies X and dissatisfies A.*

2.2 LP$^+$ validity

The multiple-conclusion generalization of LP is the standard multiple-conclusion idea: namely, expand the relation of validity from $\wp(S) \times S$ to $\wp(S) \times \wp(S)$.

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[3]The notion of an x-variant is the standard one: ν' is an x-variant of ν just if ν' differs from ν at most on x.

[4]NB: It is here and only here – viz, *(dis-) satisfaction* – where K3 model theory differs from LP. In K3, satisfaction and dissatisfaction are defined in terms of $|A|_\nu = 1$ and $|A|_\nu \in \{.5, 0\}$, respectively.

[5]Here, we let S comprise all *sentences* of the language, and count validity as a relation from sets of sentences to sentences (not open sentences).

Definition (LP$^+$). *$X \models^+_{\text{lp}} Y$ iff no LP model satisfies X and dissatisfies Y.*

An important observation for present purposes is a notable relation between invalidities in LP and corresponding validities in LP$^+$. Example: while we have the notable *in*validity

$$\exists y Hy, \forall z(\neg Hz \lor Gz) \not\models^+_{\text{lp}} \exists y Gy$$

we also have the corresponding LP$^+$ *validity*:

$$\exists y Hy, \forall z(\neg Hz \lor Gz) \models^+_{\text{lp}} \exists y Gy, \exists x(Hx \land \neg Hx)$$

It is this pattern of inconsistent-conclusion claims (or, in K3, 'complete-premise claims') that our target result captures – a 'collapse' of LP$^+$ to classical logic.

3   The classical collapse: LP$^+$

The proof has the structure of the proof in the propositional case [7]; the key difference is defining the appropriate 'inconsistency set', which is done by stacking existential quantifiers (dually, universal quantifiers for the K3 case).

Definition. *Let $\sigma(X)$ be the set of all subformulæ (including formulæ) in X.*

Definition. *Let $\alpha(X)$ be the set of all atomic formulæ $Pt_0 t_1 \ldots t_n$ in $\sigma(X)$, where the t_i are any terms.*

Definition (Inconsistency Set). *Let the v_i be object variables and t_i any terms. Then $\iota(X)$ is the set of all formulæ of the form*

$$\exists v_0 \exists v_1 \ldots \exists v_n (P v_0 v_1 \ldots v_n \land \neg P v_0 v_1 \ldots v_n)$$

for all formulæ $Pt_0 t_1 \ldots t_n$ in $\alpha(X)$.[6]

Definition (Classical model for wff). *Let A be an arbitrary formula (possibly with free variables). An LP model M is* classical *for A just if $|A|_\nu \in \{0, 1\}$; and M is classical for X just if classical for all elements of X. (If M is not classical for X, we say that M is* non-classical *for X.)*

Lemma 1. *If an LP model M is non-classical for X, then M does not dissatisfy X. (Proof: M assigns .5 to – and hence satisfies – something in X.)*

Definition (Purely classical model). *We say that an LP model is* purely classical *iff it is classical for all wff (i.e., iff it is a standard classical model).*

LP model theory gives us a new (non-classical) way of satisfying sentences but no novel *dis*satisfaction. This feature is reflected in the following lemma.

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[6]We include the case where P is a 0-ary predicate in $\alpha(X)$, in which case there are no free variables, and so no existential quantifiers. This is the case established in [7].

Lemma 2 (Classical twins)**.** *Let M be an LP model. If M is classical for X, then there's a purely classical model M' for X such that for any sentence A in X, M (dis-) satisfies A iff M' (dis-) satisfies A.*

Proof. The proof constructs (or gives the recipe for constructing) an LP model $M^c = \langle I^c, \nu \rangle$ to serve as M's 'purely classical twin'. Denotation remains per M

$$\delta^c(t) = \delta(t)$$

but we fix M^c's treatment of predicates thus:

$$I^c(P^n)(\langle x_1, \ldots, x_n \rangle) = \begin{cases} 1 & \text{if } I(P^n)(\langle x_1, \ldots, x_n \rangle) \in \{1, .5\}, \\ 0 & \text{otherwise.} \end{cases}$$

Taking the constructed 'twin' to be the theorem's target purely classical model, the proof proceeds by induction on the structure of formulæ. (Exercise.) $\square$

Definition (ν-variant model)**.** *Let M and M^* be LP models. M^* is a ν-variant of M just if the two models are exactly alike except perhaps for their variable assignments.*

Lemma 3. *Let M be an LP model; and let M be non-classical for A (i.e., assigns the value .5 to A). Then there's some atomic subformula $Pt_0 t_1 \ldots t_n$ of A for which M^* is non-classical, where M^* is some ν-variant of M.[7] (Proof is via induction, and left for reader.)*

Let $\vDash_c^+$ be the multiple-conclusion generalization of classical consequence (defined model-theoretically per above).

Theorem 1 (LP$^+$ collapse)**.** $X \vDash_{\mathrm{c}}^+ Y$ *iff* $X \vDash_{\mathrm{lp}}^+ Y \cup \iota(X)$.

Proof. The RLD follows from (elementary) reasons given in [7].

LRD. Suppose that $X \vDash_{\mathrm{c}}^+ Y$ and let M be an LP model that satisfies X.

CASE 1. M is classical for X. There are two relevant subcases for Y.

1. M is non-classical for Y. By Lemma 1, M does not dissatisfy Y, and hence does not dissatisfy $Y \cup \iota(X)$.

2. M is classical for Y. In this case, M is classical for $X \cup Y$. By Lemma 2, there's a purely classical model M' for $X \cup Y$ that agrees with M on the (dis-) satisfaction status of each element of $X \cup Y$. Since M satisfies X, so too does M'. But, then, M' does not dissatisfy Y, since $X \vDash_{\mathrm{c}}^+ Y$; and so M does not dissatisfy Y; and hence M does not dissatisfy $Y \cup \iota(X)$.

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[7]Thanks to Joshua Schechter for spotting an error in an earlier version of this lemma.

CASE 2. M is non-classical for X. Then, by definition, there's some A in X for which M is non-classical. Hence, by Lemma 3, there's some subformula $Pt_0t_1 \ldots t_n$ of A for which some ν-variant M^* of M is non-classical. But, then, by LP model theory (viz., clause for existentials),[8] M itself is at least non-classical for the sentence

$$\exists x \, \exists y \, \ldots \, \exists z \, (Pxy \ldots z \wedge \neg Pxy \ldots z)$$

and so M does not dissatisfy $\iota(X)$, and so does not dissatisfy $Y \cup \iota(X)$. □

4 The classical collapse: K3$^+$

In [7] I noted that K3$^+$ enjoys a dual result but did not give the result explicitly. I pause here to explicitly record the result.

Definition (Completeness Set). *Let the v_i be object variables and t_i any terms. Then $e(X)$ is the set of all formulæ of the form*

$$\forall v_0 \, \forall v_1 \, \ldots \, \forall v_n \, (Pv_0v_1 \ldots v_n \vee \neg Pv_0v_1 \ldots v_n)$$

for all formulæ $Pt_0t_1 \ldots t_n$ in $\alpha(X)$.

Just as $\iota(X)$ is the 'inconsistency set' for X, we have $e(X)$ the dual: it is the 'completeness set' or 'exhaustive set' (so to speak) for X.

Theorem 2 (K3$^+$ collapse). $X \models^+_{\mathrm{c}} Y$ *iff* $e(Y) \cup X \models^+_{\mathrm{k3}} Y$.

Proof. This follows the proof for LP$^+$ exactly, mutatis mutandis. □

Remark. Just as LP model theory gives us a new way of satisfying sentences but no new ways of *dis*satisfying sentences, K3 model theory delivers no new way of satisfying sentences but does give a new way of *dis*satisfying sentences. Hence, among the required changes to get the proof for K3$^+$ is Lemma 1: if a K3 model M is non-classical for X, *it does not satisfy X*.

5 The classical collapse: FDE$^+$

Putting the foregoing collapse results together gives the corresponding collapse result for the logic FDE [1, 2], which has both LP and K3 as proper extensions: it is both 'paracomplete' and 'paraconsistent', affording both a new way of dissatisfying sentences and a new way of satisfying sentences. Where $e(Y)$ and $\iota(X)$ are as above, the result is a straightforward combination:

Theorem 3 (FDE$^+$ collapse). $X \models^+_{\mathrm{c}} Y$ *iff* $e(Y) \cup X \models^+_{\mathrm{fde}} Y \cup \iota(X)$.

The proof is left as exercise.

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[8] As an anonymous referee noted, the following step also assumes, in addition to the clause for existentials, the fact – sometimes called 'agreement property' – that if models differ only on variables not occurring in A, they agree on the value of A.

6   Philosophical interest

The foregoing results explicitly record the sense in which LP and, dually, K3 (and, more generally, FDE) 'collapse' to classical logic. One might wonder whether there is any philosophical interest in these results. The answer, as suggested in [7], is affirmative, though I only sketch the idea here.

LP and K3 enjoy a good deal of philosophical interest: they are natural candidates for underwriting theories of paradoxical phenomena [3, 4, 6, 13, 14, 15, 17, 19, 23, 27]. But these logics have an apparently big defect: they are very weak. In LP, for example, material detachment (henceforth, detachment) – similarly, disjunctive syllogism – fails, where $A \supset B$ is defined as $\neg A \vee B$:[9]

$$A, A \supset B \nvDash_{\text{lp}} B$$

And there's nothing (at least nothing obvious) one can add to the premise set to remedy the situation – at least if the resulting language is to remain safe from the sorts of paradox that motivate the weaker logic to begin with. The question has always been: what to do about such weakness?

The foregoing 'collapse results', especially in the multiple-conclusion setting, nicely illustrate a response to the weakness of the given logics. Consider, in particular, the LP^+ case. While detachment fails, we nonetheless have the following validity, the cousin of detachment:

$$A, A \supset B \vDash_{\text{lp}}^{+} B, A \wedge \neg A$$

Notice that the premise set fails to imply any proper subset of the conclusion set. One way of thinking about what's going on is that logic has left us with a 'choice'. When we ask logic what follows from $\{A, A \supset B\}$, logic tells us that $\{B, A \wedge \neg A\}$ follows, not that B follows. But how do we choose between B and $A \wedge \neg A$? Since logic has left us with the choice, we are left to rely on *extra-logical* principles to make our choice (e.g., principles about rationally rejecting contradictions or etc.).

The case with K3, and more generally, FDE, is precisely the same. The logic itself is weak in various respects; however, by relying on extra-logical principles of acceptance (e.g., that in, say, physics we should accept all instances of excluded middle or etc.) we overcome the weakness of logic in our *application* of the logic – our acts of inference from choices that logic leaves us [8, 16].

6.1   Theory expansion

The philosophical application of these ideas can be seen from a different, though closely related, perspective.[10] Think about theory expansion, where theories, in this context, are sets of sentences. Theory expansion is often achieved via closure

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[9]Indeed, in LP there is no (non-trivial) detachable connective at all [11].

[10]I'm grateful to an anonymous referee for prompting this expanded discussion of some of the philosophical ideas around the collapse results.

operators (more below); but this can, in some cases, leave more expanding to do – at no fault to the closure operator. Let me (briefly) explain.

In our efforts to expand our theories, we construct closure operators under which we expand (by closing) our theories. In most cases, logic itself is insufficient as a closure operator, usually because it's silent on the non-logical vocabulary of the theory. This is why closure-operator construction often utilizes non-logical rules: rules that are not delivered by logic but are motivated by the theory's phenomena.[11]

The LP closure operator, constructed by constraining the LP$^+$ operator to singleton conclusions, is this:

$$Cn(X) = \{A : X \models_{\text{lp}} A\}$$

As with other subclassical closure operators, the problem with the LP closure operator is its weakness. In many cases, we want B to be in the expansion of our theory $\{A, A \supset B\}$, but the LP closure of $\{A, A \supset B\}$ doesn't contain B. That's just the failure of detachment.

6.2 Expansion via shrieking

One natural response to the problem is to strengthen the logical closure operator with non-logical rules. The most natural approach is the 'shrieking approach' [9, 10, 23]. The basic idea can be conceived as follows.[12] Logic (let us say, LP) dictates a wide class of models – the class of models deemed 'logically possible' according to logic. Closing our theories under logic (i.e., the logical closure operator defined over the given class of models) takes our theories as far as logic itself goes; but we might have theoretical reasons to close our theories under a stronger-than-logic closure operator. We might, in other words, have theory-specific reasons to invoke a closure operator that properly extends the purely logical closure operator (i.e., if the logical operator puts A in the closed theory, so does the stronger closure operator, but the latter puts more into the theory than the purely logical operator does). And this, at least on one (model-theoretic) way of thinking, drives the basic 'shrieking method'.

Letting $A!$ (pronounced 'A shriek') be $A \wedge \neg A$, we construct *non-logical* 'shriek rules' for given predicates in the language of the theory.[13] For simplicity,

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[11]Terminology is nonuniform (at best) around this topic. Some researchers talk of the 'logic of such-n-so' (e.g., logic of knowledge, logic of necessity, etc.), where the such-n-so is a notion or operator that is beyond bare logic on traditional criteria (e.g., 'topic neutrality', or etc.). In my view, what researchers are doing when they're doing so-called logic of knowledge (to take one example) is coming up with non-logical rules that are thought to be essential to the right closure operator for the theory of knowledge. Logic doesn't tell you that knowledge 'delivers' truth; however, the appropriate closure operator for the theory of knowledge (say, K) should deliver as much via a standard (non-logical) rule: $K(A)$ delivers A, where the *delivers* relation is simply *delivers according to the closure operator*.

[12]NB: my aim here is not to give a full discussion (or details) of how the shrieking method works; I give it only as an example that helps to illustrate some of the philosophical issues addressed by the 'collapse results' discussed in this discussion.

[13]Priest [23] advanced (what I call) the shrieking idea early on, though his formulation

consider a unary predicate P in the language. Letting $\vdash_{\mathrm{T}}$ be the theory's closure operator, predicate P's shriek rule is a non-logical rule – part of the constructed closure operator – of the form:[14]

$$\exists x\,(Px\,!)\,\vdash_{\mathrm{T}}\,\bot$$

Such rules, conceived model-theoretically, have the effect of restricting the class of LP models. Indeed, given the classical collapse result for LP^{+}, as in §3, it is not difficult to see that such shriek rules, if applied to all predicates of the theory's language, result in an in-effect classical theory:[15] the resulting theory is (negation-) inconsistent on pain of triviality. For just this reason, if we have 'shrieked' (given shriek rules for) all predicates in A, then we get the effect of detachment via our theory's (stronger-than-logic) closure operator: $\{A, A \supset B\}$, according to the bolstered closure operator, 'delivers' B for all A and B, since (we're assuming) we have shrieked A itself (and all predicates in it):

$$A!\,\vdash_{\mathrm{T}}\,\bot$$

The models recognized by $\vdash_{\mathrm{T}}$ are (for lack of better terminology) T-admissible models. Any T-admissible model in which $A!$ is true is the trivial model; and any T-admissible model in which $\{A, A \supset B\}$ is true is one in which B is true. This isn't the *validity* of detachment; but it is a sort of 'detachment' – theory-specific detachment, tied specifically to a theory's closure operator.

6.3   When shrieking ends: extra-logical choices

But now a question emerges: if the 'shrieking method' suffices for getting suitable closure operators for our theories, ones that deliver the effect of detachment, then why do we need to talk about 'choices that logic leaves us' and 'extra-logical principles of acceptance/rejection' and the like? More crudely: why not just say that detachment (or disjunctive syllogism, etc.) is *logically* invalid but that it is 'good by the lights of our bolstered closure operator'? If, in response to the weakness of our logic (e.g., failure of detachment or of disjunctive syllogism, etc.) we construct a stronger-than-logic closure operator that delivers all of the effects of having detachment, then we haven't really lost detachment or disjunctive syllogism or the like at all – and so the issue seems to be little more than terminological. But it's not so simple.

In our efforts to expand our theories, we invoke logic's closure operator. The result of such closure delivers fewer claims than we think our theory should

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suffers from using a logical-strength conditional (and his formulation is more coarse-grained than the predicate-tied approach I've advanced elsewhere). See [9] for discussion, and [10] for the basic method.

[14]In what follows, $\bot$ is some sentence that, according to either logic or the bolstered (theory-specific) closure operator, implies all sentences.

[15]By *classical theory*, in this context, I mean a theory closed under classical logic or under any closure operator that, via non-logical rules, strengthens – but assumes as basic – the classical closure operator.

contain. In turn, we bolster logic with non-logical rules in our effort to construct a more appropriate, stronger closure operator. In the case under discussion, namely, LP-based theories, we construct stronger operators via non-logical shriek rules (among other non-logical rules).[16] But, of course, we cannot forget the phenomena that drove us 'below' classical logic to begin with, such as the paradoxes (e.g., truth-theoretic paradoxes, or the like)! Accordingly, we cannot shriek *all* predicates of the language: we can't shriek the ones that deliver gluts!

Where does this leave us? The answer points back to the importance of extra-logical principles of acceptance/rejection etc. Consider, in particular, the predicates that we do not shriek. Let G be such a predicate. Since we do not have shriek rules for G (or anything with their effect), we have nothing in the closure operator of our theory beyond what basic logic delivers. And now we are back to the need for extra-logical principles in the face of 'choices' that logic gives us. In short, we want to expand our (say, super theory of) theory $\{Gb, Gb \supset Pb\}$, where P can be shrieked or not, and b is some name. We have no special non-logical rules governing G, and so our theory's overall closure operator simply looks to logic for what follows. Logic doesn't sanction detachment, and so we don't get Pb from logic. What logic does sanction is the cousin of detachment:

$$\{Gb, Gb \supset Pb\} \models^+_{\text{lp}} \{Pb, Gb!\}$$

But our aim is to expand our theory. Logic gives us the choice between Pb and $Gb!$, but it fails to zero in on exactly one of them. The rational route towards expansion is the familiar one: we now rely on extra-logical principles of acceptance and/or rejection. In the current case, we rely on a longstanding rejection principle

IR. Reject contradictions (i.e., sentences of the form $A \wedge \neg A$)!

Relying on this principle, we reject $Gb!$ and expand our theory with Pb. This is something we do; our closure operator is not up to the task.

Bolstering closure operators via (non-logical) shriek rules goes a long way towards living without detachment, disjunctive syllogism, or the like; but not every predicate can be shrieked – the paradoxical phenomena that motivated going subclassical can't be shrieked. When the shrieking stops, we are left with the choices that logic leaves us; and that's where, I have suggested, extra-logical principles come into play.

There is much more to be said on this topic, but the general philosophical interest in the given 'collapse results' is (I hope) clear. What these results suggest is that, when non-logical (say, shriek) rules are inappropriate, the weakness of the logics (or closure operators built on top of them) is overcome via other resources: we rely on extra-logical resources to *reject all elements of $\iota(X)$*

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[16]All of these claims apply to the other target subclassical cases, though in the case of K3$^+$ one adds non-logical 'exhaustion' rules or axioms. (One doesn't have to add shriek rules to K3$^+$-based closure operators, since shrieking is already part of the logic itself.) I focus on LP.

or, dually, *accept all elements of* $e(X)$, and in so doing 'return' to patterns of classical inference.[17]

Appendix: a sequent system for LP$^+$

An adequate 2-sided sequent system for (first-order) LP$^+$ may be achieved via the flip-tableau method followed in [7, Appendix]. In this appendix I simply set out the system (with a note on adding identity), rehearsing a lot of presentation from [7], and relying on the adequate first-order LP tableau system (and proofs) available in [24].[18] I note here, again, that Arnon Avron [5] was the first to record the propositional fragment of this system (and many related systems).

.1 Notation

Throughout, A and B are any sentences unless otherwise noted; Γ, Δ, Π and Σ are any sets (not multisets) of sentences; and, following convention, the comma is union and 'Γ, A' abbreviates '$\Gamma \cup \{A\}$'. I use the turnstile for sequents. In the quantifier rules, v is any variable; c is any (closed) term (constant, since we are

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[17]This idea is not at all incompatible with ideas in the literature [6, 12, 15, 23], but it does cast a new light on available resources for non-classical theorists (e.g., non-classical truth theorists, etc.).

[18]As noted in [7, Appendix], the idea of this method is to take an adequate (so-called tagged) tableau system for a given many-valued logic and 'flip' its rules to get the corresponding sequent rules – following the policy of 'positive tags on the left' and 'negative tags on the right'. In what follows, I stick to the tableau system in [24] and its (cut-free) adequacy results. (I do not explicitly formulate the K3$^+$ system, but dualizing the rules – keeping the policy about tags as above – will suffice.)

ignoring function signs); $A(v/c)$ is the result of replacing all free occurrences of v in A with c; and $[\mathbf{c}]$ is any 'new' term (standardly defined).

.2 Axioms

A1. Identity: $\Gamma, A \vdash A, \Delta$, where A is any sentence.[19]

A2. Exhaustion: $\Gamma \vdash A, \neg A, \Delta$, where A is any atomic.[20]

.3 Operational Rules

What is peculiar about negation in LP$^+$ is its interaction with other connectives. Classical rules are fine for conjunction and disjunction; it's in negation's interaction with such connectives where the non-classicality emerges. All of this is reflected directly in the familiar tableau system(s) for LP [24]; and the rules below simply rewrite such tableau rules in 2-sided set-set sequent form.

.3.1 Classical $\wedge$ rules

$$\wedge \text{ Left: } \frac{\Gamma, A, B \ \vdash \ \Delta}{\Gamma, A \wedge B \ \vdash \ \Delta} \qquad \wedge \text{ Right: } \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

.3.2 Classical $\vee$ rules

$$\vee \text{ Left: } \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \qquad \vee \text{ Right: } \frac{\Gamma \ \vdash \ \Delta, A, B}{\Gamma \ \vdash \ \Delta, A \vee B}$$

.3.3 Negated conjunctions

$$\neg\wedge \text{ Left: } \frac{\Gamma, \neg A \vee \neg B \ \vdash \ \Delta}{\Gamma, \neg(A \wedge B) \ \vdash \ \Delta} \qquad \neg\wedge \text{ Right: } \frac{\Gamma \ \vdash \ \neg A \vee \neg B, \Delta}{\Gamma \ \vdash \ \neg(A \wedge B), \Delta}$$

.3.4 Negated disjunctions

$$\neg\vee \text{ Left: } \frac{\Gamma, \neg A \wedge \neg B \ \vdash \ \Delta}{\Gamma, \neg(A \vee B) \ \vdash \ \Delta} \qquad \neg\vee \text{ Right: } \frac{\Gamma \ \vdash \ \neg A \wedge \neg B, \Delta}{\Gamma \ \vdash \ \neg(A \vee B), \Delta}$$

.3.5 Negated negations

$$\neg\neg \text{ Left: } \frac{\Gamma, A \ \vdash \ \Delta}{\Gamma, \neg\neg A \ \vdash \ \Delta} \qquad \neg\neg \text{ Right: } \frac{\Gamma \ \vdash \ A, \Delta}{\Gamma \ \vdash \ \neg\neg A, \Delta}$$

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[19]An alternative approach is to formulate Identity for *all literals*, and show that it holds for all sentences; however, a direct 'translation' of the target tableau system [24], on whose adequacy results I rely, takes Identity for *all* sentences as primitive. (NB: that one needs to take it as primitive at least for all literals is a feature of the non-classical negation at work.)

[20]A negation $\neg$ connective is sometimes said to be *exhaustive* just when its version of excluded middle holds, that is, just when $A \vee \neg A$ is valid. The role of our exhaustion axiom here is to ensure LP$^+$'s exhaustive negation.

.3.6 Classical ∀ rules

$$\forall \text{ Left: } \frac{\Gamma, A(v/c) \;\vdash\; \Delta}{\Gamma, \forall v A \;\vdash\; \Delta} \qquad \forall \text{ Right: } \frac{\Gamma \;\vdash\; A(v/[\mathbf{c}]), \Delta}{\Gamma \;\vdash\; \forall v A, \Delta}$$

.3.7 Classical ∃ rules

$$\exists \text{ Left: } \frac{\Gamma, A(v/[\mathbf{c}]) \;\vdash\; \Delta}{\Gamma, \exists v A \;\vdash\; \Delta} \qquad \exists \text{ Right: } \frac{\Gamma \;\vdash\; A(v/c), \Delta}{\Gamma \;\vdash\; \exists v A, \Delta}$$

.3.8 Negated universals

$$\neg\forall \text{ Left: } \frac{\Gamma, \exists v \neg A \;\vdash\; \Delta}{\Gamma, \neg\forall v A \;\vdash\; \Delta} \qquad \neg\forall \text{ Right: } \frac{\Gamma \;\vdash\; \exists v \neg A, \Delta}{\Gamma \;\vdash\; \neg\forall v A, \Delta}$$

.3.9 Negated existentials

$$\neg\exists \text{ Left: } \frac{\Gamma, \forall v \neg A \;\vdash\; \Delta}{\Gamma, \neg\exists v A \;\vdash\; \Delta} \qquad \neg\exists \text{ Right: } \frac{\Gamma \;\vdash\; \forall v \neg A, \Delta}{\Gamma \;\vdash\; \neg\exists v A, \Delta}$$

.4 Structural rules

Since we're using sets, we rely on the (free) rules of contraction and permutation. Cut, which is eliminable (but see §.7), is a rule:

$$\text{Cut: } \frac{\Gamma \vdash \Delta, A \qquad A, \Sigma \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi}$$

Weakening rules, namely,

$$\text{Weakening Left: } \frac{\Gamma \;\vdash\; \Delta}{\Gamma, A \;\vdash\; \Delta} \qquad \text{Weakening Right: } \frac{\Gamma \;\vdash\; \Delta}{\Gamma \;\vdash\; A, \Delta}$$

are both eliminable for a standard reason: the 'nature' of our sequents—the axioms governing them—already allow side premises (antecedents) and side consequents (succeedents).

.5 Validity.

We say that sequent $\Gamma \vdash \Delta$ is *valid* just if derivable via the above rules.

.6 Some results.

Theorem 4 (Adequacy). $\Gamma \models^{+}_{\text{lp}} \Delta$ *if and only if* $\Gamma \vdash \Delta$ *is valid.*

Proof. The soundness proof is straightforward. The completeness proof covered by Priest [24] for the corresponding tableau system carries over directly, where, as above, the negative tableau tag corresponds to the right position in our sequents, and the positive the left.[21] □

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[21]The only fiddle one needs to do is translate my talk of models (dis-) satisfying formulæ and sets of formulæ into Priest's use of so-called 'relational models'. The 'translation manual' in [7, Appendix] will serve to give the basic idea.

Theorem 5 (Cut Elimination)**.** *Any valid sequent derivable with Cut is derivable without cut.*

Proof. The given completeness proof is Cut-free, which, together with soundness, affords a straightforward induction on proof length. □

.7   Adding identity to LP

Adding identity to LP (similarly K3, FDE) is relatively straightforward, at least model-theoretically. For LP 'semantics', we ensure that all 'identity pairs' $\langle o, o \rangle$ are in the extension of the identity predicate; however, unlike classical logic (and K3), identity claims can be false too: $\neg(t = t)$ can be true (e.g., if in a model the intension of the identity predicate maps $\langle \delta(t), \delta(t) \rangle$ to the non-classical value).

But adding identity in a sequent setting raises issues highlighted by Negri & von Plato [21], issues brought to my attention by David Ripley (correspondence) and anonymous referees.[22] In general, as Negri & von Plato [21] illustrate, adding axioms for (say) identity in a sequent setting ruins either the no-exceptions cut-elimination property (all cuts, including those on identity claims, can be eliminated) or the no-exceptions subformula property (holds for all sequents, including identity-involving ones).

In our LP^+ case, we lose the general (i.e., no-exceptions) subformula property (one has the subformula property *except* for identity claims); but we keep general (no-exceptions) cut-elimination. The most straightforward 'translation' of our target tableau identity rules delivers sequent rules that preserve cut-elimination, and indeed take a form in the family of Negri & von Plato's strategies for preserving general cut-elimination [21].

Following our 'flipped-tableau translation' strategy, we augment the sequent system with identity by adding two (left) rules and a 'drop' rule:[23]

Identity rules

$$= \text{Left-1:} \quad \frac{\Gamma, A(v/c) \ \vdash \ \Delta}{\Gamma, c = c', A(v/c') \ \vdash \ \Delta} \qquad = \text{Left-2:} \quad \frac{\Gamma \ \vdash \ A(v/c), \Delta}{\Gamma, c = c' \ \vdash \ A(v/c'), \Delta}$$

---

[22]I have benefited greatly from correspondence with David Ripley on this issue.

[23]The two left rules are straightforward 'translations' of the corresponding tableau rules:

$$
\begin{array}{cc}
c = c', + & c = c', + \\
A(v/c), + & A(v/c), \text{-} \\
\downarrow & \downarrow \\
A(v/c'), + & A(v/c'), \text{-}
\end{array}
$$

The 'drop' rule, in turn, 'translates' the tableau rule which, in effect, says that one gets any free identity claim (positively marked) from nothing on any branch of the tableau, namely:

$$
\begin{array}{c}
\downarrow \\
c = c, +
\end{array}
$$

This rule 'translates' into a sequent rule that breaks the general subformula property: it drops an identity claim into nothing – the claim disappears (so to speak). The terminology of 'drop rule' is from Ripley [25, 26].

Identity drop rule

$$= \text{Drop:} \ \frac{\Gamma, t = t \ \vdash \ \Delta}{\Gamma \ \vdash \ \Delta}$$

The adequacy results in [24] cover the resulting system, which, as above, does not have the no-exception subformula property but does enjoy general (i.e., no-exceptions) cut elimination.

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