Finding tolerance without gluts

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Zach Weber [31] advances a glutty (dialetheic) approach to the sorites: the truth about the penumbral region of a soritical series is inconsistent. Joining Weber, advocating essentially the same approach, are Mark Colyvan [12] and Graham Priest [21]. The major benefit of a glut-based approach is maintaining the truth of all sorites premises while nonetheless avoiding, in a principled fashion, the absurdity of the sorites conclusion(s). I agree that this is a major virtue of the target glutty approach; however, I think that it can be had without gluts. If correct, this result weighs heavily against the proposed glutty approach, at least given the default-consistency principle that all target glutty philosophers accept: posit gluts only if there's no consistent theory that enjoys the same virtues as the would-be glutty solution.[1]

The structure of the discussion is as follows. §1 frames the sorites in an abstract way that, at least on the glutty proposal, captures the essence of the sorites. §2 and its subsections give the heart of the alternative non-glutty proposal, with §3 giving a brief recap. §4 replies to a few likely objections, and §5 offers a brief summary.

1 The sorites and the glutty solution

According to the glutty philosophers—though here, Priest [21] is most explicit, with Weber [31] straightforwardly implicit—the basic structure of sorites arguments involves some sort of equivalence-like relation that holds among all

[1]Terminology: there are many paraconsistent logics, but LP, first advanced by Asenjo under a different name [1], and later independently advanced and widely applied by Priest [16], is the chief focus here. Throughout, when I speak of 'the glutty solution' or 'the glutty philosophers' I mean the LP-based solution put forward by Weber [31], Colyvan [12], and Priest [21]. (Ripley's view [23] is in line with the target glutty view, but espoused for very different reasons that I cannot address here.) Of course, Hyde [15] is famous for discussion of a paraconsistent (in particular, subvaluational) solution to the sorites, and Cobreros et al [11] offer a non-transitive-logic approach (where Cut fails); however, my remarks are not directed towards these accounts, which have very different features from the target LP-based account. (An approach counts as LP-based if its boolean connectives—negation, conjunction, disjunction—are per the logic LP, with standard structural rules in place.)

relevant pairs of sentences in a sorites argument (e.g., between Pa_i and Pa_{i+1}, for all relevant adjacent pairs). This is where essential tolerance is to be found, and, according to the glutty theorists, is where the need for gluts emerges.

For terminology, let us say that a binary connective $\odot$ is a *pairwise-equivalence* connective just if each of the following holds, where $\vdash$ is logical consequence.

* Reflexivity: $\vdash A \odot A$.
* Symmetry: $A \odot B \vdash B \odot A$.
* *Non*-transitivity: $A \odot B, B \odot C \nvdash A \odot C$.

The last feature—non-transitivity—distinguishes pairwise-equivalence connectives from equivalence connectives, where the latter are defined in the obvious way, requiring transitivity.

The basic structure of sorites arguments, then, has the following form, where $\equiv_x$ is a pairwise-equivalence connective:

1. Pa_0
2. $\neg Pa_n$
3. $Pa_k \equiv_x Pa_{k+1}$ (for $1 \leq k < n$)

We can call the given pairwise-equivalence connective (whatever it may be) a *sorites-equivalence connective*. The question, of course, concerns the given sorites-equivalence connective: how is it to be understood in such a way that the premises of soritical arguments are coherently maintained?

Here is where the heart of the glutty proposal emerges. In short, the glutty theory maintains the truth of all premises (above) in a natural fashion: just let $\equiv_x$ be the LP biconditional $A \equiv_{lp} B$ defined as $(\neg A \vee B) \wedge (\neg B \vee A)$.[2] This affords the truth of all sorites premises, but avoids absurdity because it fails to detach: B does not follow in LP from A together with $A \equiv_{lp} B$.

The upshot: unless non-glutty solutions can keep all premises in an equally natural fashion—including, most importantly, a suitable sorites-equivalence connective—the glutty solution should be endorsed.[3]

[2]The LP framework has set $V = \{1, .5, 0\}$ of semantic values, with $D = \{1, .5\}$ the set of designated values (in terms of which validity is defined). The clauses on the boolean connectives are as per classical—or the LP-dual Strong Kleene—clauses, where interpretations $v : S \longrightarrow V$ are (total) maps from the sentences into V. The clauses run:

- $v(\neg A) = 1 - v(A)$
- $v(A \wedge B) = \min\{v(A), v(B)\}$
- $v(A \vee B) = \max\{v(A), v(B)\}$

For fuller discussion of this (and related) frameworks, see [4] or [20] or [22].

[3]I'm grateful to Crispin Wright, who, in conversation, suggested that the main attraction of the glut theory is best seen along the foregoing lines. As above, the foregoing argument is explicit in Priest's discussion, and clearly implicit in Weber's and, I think, Colyvan's paper.

I note here that all three glutty philosophers (viz., Colyvan, Priest, Weber) employ Priest's

2 Towards non-glutty tolerance

The question is whether the virtues of the glutty solution are achievable without gluts. I suggest that the answer is affirmative. To see this, back up and reflect—in abstract—on so-called 'material conditionals', which are at the heart of the glutty solution.

In bare form, the essence of a 'material conditional' invokes some operator γ that, in normal cases, is 'opposed to truth' (in some sense). In turn, the heart of a 'material conditional' defines a binary connective in terms of γ and disjunction:

$$A \supset B \;:=\; \gamma(A) \vee B$$

In turn, a 'material *bi*conditional' $\equiv_\gamma$ is defined via conjunction:

$$A \equiv_\gamma B \;:=\; (A \supset B) \wedge (B \supset A)$$

Whether this delivers a sorites-equivalence connective depends on the behavior of γ.[4] In the LP case, γ has the following key features (where the turnstile is used for validity):

- Exhaustion: $B \vdash A \vee \gamma(A)$
- Non-exclusion: $A \wedge \gamma(A) \nvdash B$

Combined with the fact that disjunction and conjunction are commutative, the exhaustive-nonexclusive behavior of γ yields the virtues of the LP approach to the sorites—and, in particular, to the tolerance or sorites-equivalence connective.

Of course, LP theorists maintain that γ is *negation*. But—for present purposes—we may take negation to be as per classical theory (viz., exhaustive and exclusive), while seeing γ as something else. This is the approach I suggest here. The result enjoys all virtues of the LP solution *but without gluts*. And this, I take it, is strong consideration against the glutty proposal—even if, like myself, you do not ultimately think that negation is classical.[5]

so-called inclosure argument as an argument for the target glutty approach to vagueness. (From conversation I gather that the idea for this application of the inclosure scheme seems first to have come from Colyvan, prompted by discussion with James Chase.) I find this argument plainly wanting, and so concentrate on the alternative argument above. My chief reason for rejecting the inclosure-paradox strategy is briefly discussed in §4.

[4] There is, of course, much else that it turns on, from disjunction and conjunction behavior to 'deep structure' of the logic (e.g., structural rules, etc.); however, I will assume fairly standard behavior on these fronts—not only for disjunction and conjunctions, but also structural rules.

[5] While this paper is largely on behalf of classical-logic theorists, it is not just for such theorists. My own view [3] is that negation itself is along LP lines, and that there are gluts in (and only in) the purely semantic parts of language (e.g., liars, etc.); however, I see little force in the 'inclosure arguments' that purport to expand the domain of gluts. (See §4.1.)

2.1 Unassertability operator

There is a notion of assertability for which truth is necessary but not sufficient: A is assertable only if true; however, the converse need not hold. What more, in conjunction with truth, is required for the target notion(s) of assertability is not my present concern—though various epistemic conditions are good candidates. For present purposes, I simply assume some such notion, showing how it serves to provide the virtues of the glutty solution without the gluts.

The basic thought is just this: what the sorites teaches us is not that there are 'true contradictions', but rather that there are true-but-unassertable sentences. Like the glutty solution, we maintain the truth of all sorites premises, and we avoid absurdity because our tolerance or sorites-equivalence connective—defined in terms of our target (un-)assertability operator—fails to detach.

A formal sketch of the idea is given below, but the basic idea runs as follows. Let us assume a point-based framework—worlds, let us say, which are one and all negation-complete and negation-consistent. To make matters simple, let us think of our space of worlds as having a special world @, which serves as the base or actual world. In turn, think of *truth* as what's true at @.

Towards our target connective, let us posit a (total) function $\star : W \longrightarrow \wp(W)$ from worlds to sets of worlds, where $x^\star$, the value of $\star$ at world x, may be thought of as containing the worlds relevant to what is assertable at x.[6] Thinking of truth as truth at @, we may—following a familiar thought—think of assertability as truth at all assertability-relevant points, namely, all points in $@^\star$.

What constraints are imposed on the $\star$ operator? The answer depends on which notion of assertability is at hand. For present purposes, we are leaving the notion of assertability open except for the necessary condition of truth: assertable sentences are true. Accordingly, we impose the constraint that, for all points x and all sentences A, if A is assertable then A is true. In bare disjunctive form, the constraint is that, for any world x, either A is unassertable at x or A is true at x *or both*. The last option (viz., *both*) is directly relevant to the sorites—and, in particular, to the glutty philosophers' solution. But first let us put a bit more formal clarity to the idea.

2.2 A formal picture

Let us focus on the standard propositional (or boolean) level with one change: we add a primitive *un*assertability operator μ into the mix.[7] The framework is

[6] In my original thinking for this paper, I took a more directly 'star-semantical' framework, where $\star$ is an operator on W, and $x^\star$ (a world, not a set of worlds) was treated as the 'assertability record' for x. Thanks especially to David Ripley, I now rely on a simpler—and much more familiar—framework. While the original star-semantics framework [10, 25, 29] has virtues in the present context, I leave discussion for another occasion. (For star-semantics experts: take $x^\star$ to be what is assertable at x, demanding only that truth at $x^\star$ implies truth at x. Impose classical constraints on @ but only Strong Kleene constraints on $@^\star$. Define the target *un*assertability operator as untruth at the given star point.)

[7] One might wonder why we go with an *un*assertability operator instead of the 'positive' approach—assertability. The short answer is that the target application of this—as will be evident—uses the unassertability operator in place of the 'normally-opposed-to-truth operator'

a point-based framework. The boolean connectives get a standard 'extensional' treatment (i.e., truth-at-a-point conditions involve looking nowhere beyond the point of evaluation), while the unassertability operator receives an 'intensional' treatment (i.e., requires looking beyond the point of evaluation). The basic details are as follows.

Let us use W for our set of worlds, with $@ \in W$ our base world, and assume that our truth-at-a-point relation $\models$ relates each point $x \in W$ to some (proper) subset of atomics. In turn, we give truth-at-a-point conditions for boolean connectives in the following, familiar—classical—fashion.

- Negation: $x \models \neg A$ iff $x \not\models A$.
- Disjunction: $x \models A \vee B$ iff either $x \models A$ or $x \models B$.
- Conjunction: $x \models A \wedge B$ iff $x \models A$ and $x \models B$.

Unlike the connectives above, our unassertability operator μ is 'intensional'. For convenient notation, where $Y \subseteq W$, we use '$Y \models A$' to mean that $x \models A$ for all $x \in Y$, so that $Y \not\models A$ holds just when there's some $x \in Y$ such that $x \not\models A$. Then the central clause for our unassertability connective μ runs thus:

- Unassertability: $x \models \mu(A)$ iff $x^\star \not\models A$.

What constraints do we impose on the star? We impose only one constraint:

- Star constraint: for all $x \in W$, either $x^\star \not\models A$ or $x \models A$.

Finally, logical consequence (validity) may be defined over base points:

- $X \vdash A$ iff for every model, if $@ \models X$ then $@ \models A$.[8]

A few notable features of μ jump out:

- Exhaustion: $B \vdash A \vee \mu(A)$.

  Proof. For this to fail, we require that $@ \not\models A$, in which case—by the star constraint—we have that $@^\star \not\models A$, in which case, by the μ condition, we have that $@ \models \mu(A)$.

- Non-exclusion: $A \wedge \mu(A) \nvdash B$.

  Countermodel: $W = \{@, w\} = @^\star$, and $@ \models A$ but $w \not\models A$ and $@ \not\models B$.

The recipe for our target tolerance connective is now plain.

γ discussed in §2. But if one prefers, one can directly define an assertability operator α in what will be the obvious way.

[8]Here, $y \models X$ indicates that $y \models B$ for all $B \in X$.

2.3 Tolerance connective

Instead of taking negation to play the role of γ in our sorites-equivalence connective (see §2), we take γ to be μ. In particular, we define

$$A \sqsupset B \;:=\; \mu(A) \vee B$$

In turn, we define the target sorites-equivalence connective on the usual recipe:

$$A \equiv B \;:=\; (A \sqsupset B) \wedge (B \sqsupset A)$$

This is a sorites-equivalence connective (though proofs are left as exercise):

1. Reflexivity: $\vdash A \equiv A$.
2. Symmetry: $A \equiv B \vdash B \equiv A$.
3. *Non*-transitivity: $A \equiv B, B \equiv C \nvdash A \equiv C$.

Moreover—and most importantly—we have exactly what, in the glutty response, saves us from sorites-driven absurdity:

$$A, A \equiv B \nvdash B$$

Any sorites argument itself provides a counterexample to detachment. In a sorites, somewhere along the line we have an A such that $@ \models A$ and $@ \not\models B$, and this A is in the penumbral region whereat A is true-but-unassertable, that is, $A \wedge \mu(A)$ is true. But since $A \wedge \mu(A)$ is true, so too are both $\mu(A)$ and A, and hence—by Addition (i.e., every sentence implies the disjunction of itself and any sentence)—so too are both $\mu(A) \vee B$ and $A \vee \mu(B)$, and so we have the truth of both $A \sqsupset B$ and its converse $B \sqsupset A$, and so we have $@ \models A \equiv B$. Hence, putting all of this together, we have that $@ \models A$ and $@ \models A \equiv B$ but $@ \not\models B$. As in the glutty proposal, we are saved from sorites-induced triviality by the non-detachability of tolerance. But there is a difference: gluts are unnecessary.

3 All the virtues without the gluts

The glutty philosophers maintain that the LP-based response to the sorites has virtues not enjoyed by non-glutty approaches. The chief virtue is that the truth of all sorites premises is maintained—including the essential tolerance of vague predicates. What the sorites teaches us is that the tolerance of vague predicates is expressed via a non-detachable 'conditional' or, as I've put it, a non-detachable sorites-equivalence connective.

What the foregoing proposal shows is that gluts are unnecessary for purposes of enjoying the given virtues. Indeed, even in a classical setting, where negation is per classical theory, all of the virtues of the LP-based approach are available: we simply need to locate 'true tolerance' somewhere other than in a detachable conditional; we simply have to locate our target operator—the 'normally opposed to truth' operator—in something other than negation. In the context of vagueness, some sort of unassertability operator is promising, particularly one for which truth is necessary but not sufficient. The foregoing discussion models just such a course, and shows that it is achievable without gluts. I take this to be a result that significantly diminishes the promise of the glutty approach.

4 Objections and replies

The aim of this section is to address the most salient objections likely to be raised against the foregoing discussion. Constraints on space leave other objections and issues for future debate.

4.1 Objection: inclosure-paradox argument

It may be that the true-tolerance argument is insufficient for adopting a glutty (over classical) approach to the sorites. But the glutty philosophers have another argument—and, indeed, Weber explicitly endorses the additional argument.

Weber, following Priest and Colyvan, pursues Priest's uniform-solution strategy: if phenomena X and Y are, at their core, essentially the same, then they should be treated the same. In the case of paradoxes: if paradoxes X and Y are, at their core, essentially the same, then their solutions ought be the same. The glutty philosophers argue that the Liar and sorites paradoxes enjoy the same basic core, and, hence, should be treated the same—as glutty. This uniform-solution strategy for a glutty approach to the sorites is untouched by the foregoing objections to the preserving-tolerance argument.

4.2 Reply

A full-blown discussion of the relevant uniform-solution strategy is beyond the available space of this paper. Many good objections have been made to this approach, though debate continues [14, 17, 26, 18, 30]. My aim here is only to briefly sketch what I take to be the main problem with the strategy.

As the objection states, the uniform-solution strategy turns on identifying the 'core' of the target phenomena—in this case, paradoxes. The obvious question is: what is the core in the cases at hand? Here is where Priest's 'inclosure schema' [19] comes in: the core of many traditional paradoxes is the *inclosure structure* (details of which may be found in any of the target papers).

The Liar paradox is said to be an *inclosure paradox*: there is a *plausible* argument that it enjoys the inclosure structure. (Importantly, the inclosure-structure argument need not be *valid*—else there'd be no argument for the sorites counting—but it does have to be *plausible*.) Similarly, Russell's paradox (in both property- and set-theoretic versions) is an inclosure paradox: there is a *plausible* argument that it (they) enjoy the inclosure structure. So argue the target glutty philosophers.

I agree: there are plausible arguments that such paradoxes enjoy the inclosure structure. What of the Barber paradox (or the Secretary-Club paradox or etc.)? On the surface, the Barber's structure (similarly for Secretary's Club) is precisely that of Russell's paradox. But the Barber paradox—all (target) glut theorists agree—does not require a glutty treatment. (Similarly for the Secretary's Club and other variants; I focus here on the Barber.) Hence, it had better not count as an *inclosure paradox*, lest the uniform-solution principle for gluts be triggered—thereby committing us to the Barber's both shaving and not shaving himself, something to be rejected.

This is where *plausibility* becomes critical in the target strategy: to be an *inclosure paradox*—to be a phenomenon of the sort that properly triggers the uniform-solution principle for gluts—there needs to be a *plausible* inclosure-structure argument. Fortunately, in the Barber's case, any candidate for such an argument requires the (let us grant, flatly implausible) empirical premise that there is some such Barber in some such village. So, a *plausible* inclosure-structure argument for the Barber's paradox is not forthcoming—or so the target glutty philosophers maintain, and I grant.

But what of other apparently-inclosure-structure phenomena that are free of empirical assumptions—or, at least, as free of such assumptions as the Liar? The elephant in the room, of course, is Curry's paradox. In its material-conditional form, Curry's paradox is no different than a standard disjunctive Liar: *either I'm untrue or everything is true.* This (material-conditional) version of Curry's paradox counts, for the glutty philosophers, as an inclosure paradox: the inclosure-structure argument is precisely as plausible as that of the standard Liar paradox (for which we here assume a plausible inclosure-structure argument). But what, now, of Curry's paradox in a more familiar—detachable—form?[9]

Of course, any inclosure-structure argument for Curry's paradox will be deemed invalid, just as the inclosure-structure argument for the sorites is deemed invalid. The question concerns the critical notion of 'plausibility' that is supposed to tell the big difference between 'real' inclosure paradoxes and mere lookalikes.

The invalidity of Curry-related arguments points (in target cases) to so-called contraction principles; but the issue is plausibility. There may well be some binary operators $\to$ for which (contraction) rules or principles such as

$$A \to (A \to B) \therefore A \to B$$

or, indeed,

$$A \wedge (A \to B) \to B$$

are invalid; but they are certainly very *plausible* for (detachable) conditionals. At the very least, it takes a great theoretical shift to find such things

[9]In LP, there's no detachable (modus-ponens-satisfying) conditional [7], the material conditional being the chief conditional. But note that all target glut theorists assume that we have detachable conditionals in the language—indeed, important ones that do detachment work in fundamental principles such as the T-schema or property schema (naïve sets, if you like). If one were to give this up, as in my own current work [5, 6], the problem(s) with Curry's paradox to which I am pointing are reduced (if not eliminated). But, again, the target glutty philosophers embrace extra-LP detachable conditionals.

implausible.[10] That this is so is witnessed by the fact that stories about 'non-normal worlds' and 'ternary relations', which are invoked to invalidate target contraction principles, remain philosophically dubious to many open-minded philosophers—indeed, even those who are open to the possibility of gluts. But with the plausibility of such (contraction) principles, the inclosure-structure argument for Curry's paradox is straightforward (and at least as plausible as the corresponding inclosure-structure argument for the sorites).[11]

Again, it may be that contraction-like principles are ultimately invalid (just as detachment for tolerance principles is supposed, by the glutty theorists, to be ultimately invalid); but they're certainly plausible enough to provide plausible inclosure-structure arguments for Curry's paradox (no less plausible than the corresponding inclosure-structure argument for the sorites, which relies—for the transcendence condition—on detachable tolerance). And so it looks like Curry's paradox ought count as an inclosure paradox if the Liar and Russell's do—and certainly if the sorites is to count.

In the end, too much weight is required of the notion of a *plausible inclosure-structure argument* for it to do the critical work of classifying the target phenomena. One immediate (though not cure-all) remedy would be to require validity or, indeed, soundness: let *inclosure paradoxes* be those for which there is a *sound* (or, at least, *valid*) inclosure-structure argument. This immediately knocks out Curry's paradox, but it likewise knocks out the sorites—contrary to the hopes of target glutty philosophers. What other remedies may be available are unclear. But as things stand, relying on *plausibility* strikes me as plainly insufficient, since Curry's paradox would seem to count if the sorites does.[12]

Upshot: because, then, I think the overall inclosure strategy to be generally weak, I do not see it as a strong—indeed, see it as a weak—argument for a glutty solution to the sorites.

[10] So-called *structural* contraction is equally—if not more—plausible; and indeed is something enjoyed by the logics that the target glutty philosophers advance. Similarly, *conditional proof*—in the form of a simple deduction theorem—is likewise plausible; and it, together with a detachable T-schema (and other logical features that are uncontroversial among target theorists), is sufficient for Curry trouble in one form or another [8].

[11] In the detachable Curry-paradox cases, Ω and θ are just as per the Liar—respectively, the set containing (or 'exemplified by', in the property-theoretic setting) all and only the truths, and the property of being definable or nameable. As in the sorites argument, we assume that Ω is not trivial (i.e., doesn't contain all sentences), and we let $\perp$ be such a sentence not in Ω. Then δ takes subsets X of Ω and, where t_X names X, delivers sentences C of the form $\langle C\rangle \in t_X \to \perp$. (Here, following notation in target works, $\langle\,\rangle$ is a function—a naming device—from sentences to names of sentences: $\langle C\rangle$ names C.) *Transcendence* is achieved by the fact that $\perp \notin \Omega$ and the fact that, since $\to$ detaches, C (which, in the limiting case, is $\delta(\Omega)$) is in Ω only if $\perp$ is too, which can't be. *Closure* is achieved via standard Curry reasoning from one of the contraction principles and T-scheme (or comprehension scheme).

[12] I should note that even a criterion of '*a priori* plausibility', sometimes invoked by Priest [21], does not help. By my lights, if any two paradoxes look *a priori* like they're in the same family, it's the Liar and Curry's paradox: both appear to imply absurdity; both do it via truth (or exemplification, *true of*, etc.) and *usually* circularity [2, 28, 33], with the Liar going directly via negation and Curry via a (detachable) conditional. And, again, with respect to the inclosure-structure argument for the sorites: if *it* counts as suitably 'a priori plausible' (whatever the account of *a priori plausibility* may be in the end), the corresponding argument for Curry's paradox is likely to count too.

What is interesting is that the glutty solution has an alternative—and independently interesting—argument, namely, the one addressed in this paper (from maintaining the truth of tolerance principles). But that argument, as I've tried to make plain, also fails.

4.3 Objection: virtue of rejecting sharp cutoffs

Even if neither the inclosure-based argument nor the preserving-tolerance argument for gluts works, Weber has given us yet another important argument for the glutty approach.[13] In short, Weber's approach gives us an explanation for why no sharp borders are forthcoming in a sorites sequence: any successful candidate for a sharp border, anything truly counting as *the first non-*φ or a relevant φ*-cutoff* in a given sorites sequence, is not unique; there is another *first non-*φ, another φ*-cutoff*. But if there's no *unique* φ-cutoff (no unique 'first non-φ'), there's hardly a sharp border. But, then, since avoiding sharp borders has been one of the chief desiderata of solutions to the sorites, Weber's approach enjoys a distinct advantage over other theories.

4.4 Reply

The objection maintains that since Weber's proposal avoids a unique φ-cutoff, it thereby sufficiently avoids accepting that there is a sharp border. But this is too quick. Weber's approach not only accepts that there's no unique φ-cutoff; it also accepts that there *is* a unique φ-cutoff. More clearly: his theory, when closed under the given logic, contains the claim that there are unique cutoffs; indeed, for each cutoff, his theory contains the claim that it is unique.[14] (See Appendix for elaboration on this point.) Hence, the argument from avoiding unique cutoffs to avoiding sharp boundaries—and thereby achieving virtues over alternative theories—is misplaced in Weber's case. One does not get that all cutoffs are *identical*, as Weber emphasizes [31, Appendix 2]; however, one does get that each cutoff is *the unique cutoff (for given predicate)*. Instead of avoiding sharp borders, Weber's approach multiplies them: there is not just one sharp border; there are many, each *the unique cutoff*, on his theory.

4.5 Objection: other conditionals

Agreed: the classical-logic (or, generally, non-paraconsistent) theorist may enjoy the virtues of the glutty solution in the manner proposed—namely, locate 'true

[13] I am grateful to an anonymous referee for voicing this objection. My formulation of the objection sticks closely to the referee's (and is lifted verbatim at points).

[14] Weber won't contest the point, and in fact mentions it in passing in [31, §5.2]. But, as the current objection reflects (and, again, I'm grateful to a referee for pointing to this), it is important to see that Weber's theory is committed to there being a unique cutoff—indeed, that x is the unique cutoff, for any cutoff x. I make this point explicit in the Appendix.

tolerance' via *unassertability*.[15] In particular, the key operator, unlike negation, is to be treated as an *intensional* operator—one whose truth-at-a-point conditions crucially invoke other points. The result affords a sorites-equivalence operator that enjoys all the benefits of the glutty solution without gluts.

The trouble is that this solution leaves the *other material-conditional sorites* untouched: the sorites that involves the *extensional* connective defined by taking γ to be (extensional) negation is still unresolved. Adding a different connective by taking γ to be (intensional) unassertability doesn't solve the original problem.

4.6 Reply

There are two replies to make. To begin, the classical-logic (or otherwise target non-paraconsistent) theorist about vagueness accepts that the (extensional) material-conditional version of the sorites is simply valid but unsound: there's no 'true tolerance' connective involved in such versions of the sorites. The proposal is that true tolerance—the only tolerance really involved in vagueness—is something else: it's something expressed via an *intensional* connective. So, the objection does not undermine the proposed (non-paraconsistent) response.

The second reply: for present purposes, the aim is not to defend the sketched classical (or, at least, non-paraconsistent) response to vagueness; it's rather to show that the virtues of the glutty response—and I do think them virtues—may be enjoyed by the classical theorist. And along these lines, the glutty theorist is in no better situation than the classical (non-paraconsistent) theorist. Let me briefly expand on this point.

The non-paraconsistent theorist, as above, maintains that the heart of the sorites involves an intensional operator, and that the extensional (non-intensional) versions of the sorites are not the important phenomenon: they are ones where the truth of the premises (e.g., extensional versions of tolerance) are flatly implausible. The *glutty* theorist, on the other hand, maintains just the opposite: the glutty philosophers offer a solution to the extensional version but shun corresponding intensional versions (about which more below). Specifically, the glutty philosophers maintain that the heart of the sorites involves an extensional operator, and that the intensional (non-extensional) versions are not important: they are ones where the truth of the premises (e.g., intensional versions of tolerance) are flatly implausible.[16]

What are the target 'intensional versions' of the sorites that the glutty theorists find implausible? There are lots of them, but the basic thought is that

[15] Again, one can treat the given operator either as a primitive unassertability operator, as I have done here, or as definable from negation and a primitive assertion operator. Either option affords the same benefits.

[16] Note: experts know that negation in LP can be given an intensional semantics [10, 25, 29]; however, *all* connectives *can be* given such a semantics [24]. By calling LP's account of negation *extensional* I mean only that it does not essentially require a non-extensional treatment. (I admit that the distinction is not as precise as one might hope; however, the target paraconsistent theorists will not balk at the basic point that LP negation is not essentially intensional.)

'true tolerance' is expressed (in at least one direction if not both) using an intensional conditional. The glutty philosophers are aware of the issue, and each gives essentially the same argument for why these versions of the sorites is not the real one. Priest puts the reasoning thus:

> In weak relevant logics [in the vicinity of interest to the target glutty solutions] of the kind required for paraconsistent set theory and semantics, if $\alpha \rightarrow \beta$ is true, then α entails β. That is, α, on its own, is a logically sufficient condition for β. ...The major premises of sorites arguments have little plausibility if conditionality is construed in this fashion. For no h is Pa_h a *logically* sufficient condition for Pa_{h+1}. [21, pp. 73–4]

The thought, then, is that at least for familiar detachable conditionals $\rightarrow$ in our (target paraconsistent) language, it is implausible that $Pa_i \rightarrow Pa_{i+1}$ is true, since $\rightarrow$ requires too strong a relation between antecedent and consequent. Weber [31, p. 1040] is explicit on the 'intensional' strength of the non-extensional versions of sorites-equivalence connectives:

> There needs to be a genuine, intensional connection between the premiss and the conclusion for $\Psi \rightarrow \Gamma$ to hold. The connection is very strong, much more than is intended in many contexts...and not a likely candidate for what is being asserted in a sorites premiss.

These remarks, by Priest and Weber, might be right; however, they're in important ways beside the point. Instead of considering 'bare' forms of tolerance principles, we need to consider the more explicit (and fundamental ones)—the versions that, as Graff Fara [13] puts it, are properly called 'sorites premises' or, simply, *(full-dressed) tolerance principles*. The difference is that full-dressed tolerance principles wear the target tolerance relations explicitly in their antecedents. In particular, where R is a tolerance relation for P (e.g., perhaps 'x differs by a nanosecond from y', where P might be 'x is late' or the like), the bare tolerance would look like

$$Px \rightarrow Py$$

but the full-dressed tolerance looks like

$$Px \wedge Rxy \rightarrow Py$$

But once we fully dress them, the considerations that target glutty philosophers mount against the plausibility of intensional versions of tolerance—namely, too strong to be plausible—seem to disappear.

Of course, provided that, as in the target glutty logics, the logic is transitive (e.g., Cut holds), no non-trivial theory can enjoy detachable tolerance.[17] The point above is simply that the considerations put forward by glutty philosophers do not tell against the plausibility of detachable tolerance. The explanation must lie elsewhere—though where it is to be found is a matter I leave open.

[17] For options along non-transitive lines, see [11].

The important upshot: the glutty philosophers are in no better position than our classical or non-paraconsistent theorists when it comes to the need to distinguish the 'real sorites' from the 'unreal' version(s). The former theorists see *extensionality* at the heart of the sorites and intensional versions as counterfeit; and the latter theorists see the opposite.[18] Either way, if the objection of §4.5 is a problem for the proposed non-glutty solution, it is similarly a problem for the glutty one.

5 Summary

I have argued that a classical theorist—or, generally, non-paraconsistent theorist—can enjoy the essence of the LP-based glutty solution by treating γ, the critical operator at the heart of the sorites-equivalence connective, as something other than negation. Doing so affords all the virtues of the glutty solution without requiring gluts. Instead of the sorites—the penumbral region—calling for 'true falsehoods', we instead see it calling for *true unassertables* (as it were): true but unassertable sentences, not gluts. Inasmuch as a consistent solution is preferable to an inconsistent one, this result serves as a powerful argument against the glutty proposal.[19]

Postscript

I should note a parallel that might be evident to some readers, though it did not occur to me until after having written the paper. I am grateful to Michael Hughes, who, in conversation, noted the following point. (I do not belabor the point, but think it worth explicitly noting.)

I have framed the discussion in terms of a sorites-equivalence or 'tolerance' connective, and focused the discussion on the question of which operator plays the role of γ in the target connective. My suggestion points to *assertability*—or

[18] Let me make plain that, in the end, the issue need not be framed as 'intensional versus extensional' connectives. This is the way the issue is framed by Weber's discussion and Priest's discussion (see quotes above), and I have followed their way of framing things. But instead, one could take the *unassertability* proposal as locating 'true tolerance' somewhere in a different-from-negation extensional connective. Example: take LP's 'negation' to in fact be an extensional *unassertability* operator, thereby walking along lines suggested by 'analetheists' [9]. (In short: see the three semantic values of LP as *true and assertable*, *true but unassertable*, and *untrue and unassertable*.) If one took this route, the reply above would be largely the same: the glutty theorists need to distinguish 'true tolerance' from lookalikes in precisely the same way that classical-logic-based theorists do. Both parties give one reply to some sorites and another to others: valid but unsound in the lookalike-tolerance versions, but invalid-but-all-true-premises in the 'true tolerance' versions. (Thanks to a referee for prompting inclusion of this point.)

[19] In addition to very helpful anonymous referees, I am grateful to John Burgess, Roy Cook, Aaron Cotnoir, Charlie Donahue, Delia Graff Fara, Hartry Field, Patrick Greenough, Michael Hughes, Graham Priest, Greg Restall, Marcus Rossberg, Lionel Shapiro, Roy Sorensen, Achille Varzi, Zach Weber, Crispin Wright, and audiences at the NIP logic-language conference in Aberdeen, at Princeton University, and at the University of St Andrews. I owe special thanks to David Ripley for discussion, and also spotting a serious problem in an earlier version.

*un*assertability, as I put it. (Again, though, one can get one out of the other in the obvious way.) Assertability is an operator that 'releases' (i.e., has truth as a necessary condition) but does not 'capture' (i.e., truth is not sufficient for assertability).[20] But there are other operators in the vicinity.

Another familiar operator is *knowledge*, which releases but fails to capture. What Hughes noted is that one could take *knowledge* or, for perfect parallel with the approach I've advanced here, *un*knowledge to play the role of γ in the critical sorites-equivalence connective. In particular, where Υ is an appropriate *un*knowledge operator—say, defined via a knowledge operator and (let us say, classical) negation—we define

$$A \sqsupset_k B \ := \ \Upsilon(A) \vee B$$

and, in turn, the target sorites-equivalence connective

$$A \equiv_k B \ := \ (A \sqsupset B) \wedge (B \sqsupset A)$$

On a plausible account of knowledge (or, as the case may be, *un*knowledge), the connective $\equiv_k$ is reflexive, symmetric, and non-transitive—and it won't require gluts. Instead of gluts, one requires true-but-unknown (or, perhaps, unknowable) sentences to be involved in the (real) sorites. As Hughes notes, this might situate the proposal in the vicinity—at least the outskirts—of classical-logic-based epistemicists [27, 32].

My point, in this postscript, is only to flag that one needn't take the assertability route that I suggest in the paper; one could think along lines advanced by other classical-logic-based theorists. The main point remains the same: we can have the given virtues without gluts.

Appendix: on the (many) unique cutoff(s)

That Weber's theory is committed to the *uniqueness* of any cutoff may be seen as follows.[21] Focus, for simplicity, on unary (vague) predicates. (Generalizing is straightforward but tedious.) Where a_i is an object in a sorites series, an *a_i-φ-cutoff claim* is any claim of (or equivalent to) the form

$$\neg\varphi(a_i) \wedge \forall j(j < i \supset \varphi(a_j))$$

We have the following fact:

Lemma 1 (Cutoffs) *any a_i-φ-cutoff claim is true (designated) over all* and only *the following transitions: $\langle 1, .5\rangle$, $\langle .5, .5\rangle$, and $\langle .5, 0\rangle$.*

[20] The terminology is used for a common pattern: an operator φ *releases* just if $\varphi(A)$ implies A, and it *captures* just if the converse holds. Various operators only exhibit release behavior; various exhibit only capture behavior. (Though not relevant here, truth is in the unique position of playing both capture and release.)

[21] The point of this Appendix is simply to spell this out a bit more than it is in either Weber's paper or elsewhere. Again, it is not something to which Weber (or the other target glutty philosophers) will object; however, it is important to see the commitment. (Thanks, once again, to a referee for encouraging clarity on this.)

Proof: the proof assumes only two items (both involved in the glutty proposal):

1. $\langle 1, 0\rangle$ transitions are non-existent. (The whole point of the glutty proposal is that sorites transitions always involve gluts.)

2. Tolerance-ordering claims (viz., literals involving '$<$' in Weber's notation) are always evaluated classically: modeled as 1 or 0. (Changing this may afford new options, but it would be different from the target glutty proposal, which seems to take the ordering of a sorites to be a glut-free matter).

Given (1)–(2), the proof falls directly out of LP-connective behavior. □

Fact 1 (Cutoffs) Any true a_i-φ-cutoff claim is a glut.

Proof: This is immediate from the Cutoff Lemma and LP framework. □

Towards establishing the unique-cutoffs claim, first recall the definition involved in Weber's approach. Let a_i be any object in a φ-sorites series. Then a_i's being a *unique* φ-cutoff comes to the conjunction of a_i's being a φ-cutoff

$$\neg\varphi(a_i) \wedge \forall j(j < i \supset \varphi(a_j))$$

and a_i's being uniquely so

$$\forall k([\neg\varphi(a_k) \wedge \forall j(j < k \supset \varphi(a_j))] \supset a_k = a_i)$$

Given this definition, the unique-cutoffs fact falls out:

Fact 2 (Unique Cutoffs) Every φ-cutoff a_i is unique (i.e., the given unique-φ-cutoff claim with respect to a_i is true).

Proof: Let a_n be a φ-cutoff, in which case, by the Cutoffs Fact (viz., Fact 1),

$$\neg\varphi(a_n) \wedge \forall j(j < n \supset \varphi(a_j)) \tag{1}$$

is glutty. For simplicity, abbreviate (1) as $C(\varphi, a_n)$. Now, consider the relevant *unique-cutoff* claim for a_n, namely,

$$C(\varphi, a_n) \wedge \forall j\,[\neg\varphi(a_j) \wedge \forall h(h < j \supset \varphi(a_h)) \supset a_j = a_n]$$

Since $|C(\varphi, a_n)| = 0.5$,[22] the unique-cutoff claim is *untrue* (unsatisfied) only if the right conjunct has value 0. The right conjunct has value 0 iff at least one instance (of the given generalization) has value 0, iff there's some a_k such that

$$|\neg\varphi(a_k) \wedge \forall h(h < k \supset \varphi(h)) \supset a_k = a_n| = 0$$

But in LP, this can happen only if $|\neg\varphi(a_k) \wedge \forall h(h < k \supset \varphi(h))| = 1$, which is impossible given the Cutoffs Fact. □

[22] I use the bar notation, instead of the v notation in Weber's (similarly, Priest's) paper, for ease of reading.

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